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Orbit-Averaged Behavior of Magnetic Control Laws for Momentum Unloading

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Analytical formulas are derived for orbit-averaged behavior of magnetic control laws for unloading the excess angular momentum of a spacecraft reaction wheel control system in the presence of secular environmental torques. The specific example of an axially symmetric spacecraft with an inertially fixed attitude for which the dominant environmental torque is the gravity-gradient torque is treated in detail, but extensions of the general approach to other inertially fixed and Earth-pointing spacecraft are discussed. The analytical formulas are compared to detailed simulations performed for the Solar Maximum Mission spacecraft, and agreement to within 10% is found. The analytical formulas can be used in place of detailed simulations for preliminary studies, and can be used to find selected cases giving the most stringent tests of momentum unloading capability for which detailed simulations may be performed.

Introduction

MAGNETIC torquing techniques for spacecraft attitude control have been the subject of intensive investigation.[1-4] Many spacecraft have used magnetic torquing in conjunction with spin stabilization, either in a single- or dual-spin configuration. For attainment of high pointing accuracy, however, a control system employing three or more reaction wheels—or, alternatively, control moment gyros—is favored. Reaction wheel torques can be used for slewing and compensating for internal disturbance torques and periodic external disturbance torques. The secular component of external disturbance torques would lead to saturation of the momentum capacity of the reaction wheels, so either mass expulsion or magnetic control torques are needed to dump excess reaction wheel angular momentum.

Magnetic control torques have several advantages for near-Earth missions, including smoothness of application, essentially unlimited mission life (due to the absence of expendables), and absence of catastrophic failure modes.[3] It is important to note that magnetic torques need not compensate for the entire disturbance torque in such an application, but only for its secular (orbit-averaged) component. Closed-loop magnetic control for momentum unloading has been used on the Orbiting Astronomical Observatory (OAO)[5] and is proposed for the Space Telescope (ST)[6,7] and for the Solar Maximum Mission (SMM) spacecraft.[8]

The simplest momentum unloading strategy is embodied in the cross-product law[1-3,6,8]

$$M = g\Delta h \times B \tag{1}$$

where M is the commanded dipole moment, g is a control gain, Δh is the vector sum of the differences between the reaction wheel angular momenta and their nominal values, and B is the Earth's magnetic field at the position of the spacecraft. This control law is easily implemented onboard since Δh and B are easily obtained from reaction wheel tachometer and magnetometer data (assuming that stray magnetic fields are negligible or compensated in hardware or software). The dipole moment is perpendicular to B (Kamm's "efficiency condition"[1]), which is desirable since a component of M parallel to B would not produce a torque. The control torque given by Eq. (1) is

$$N_c = M \times B = -gB^2 \Delta h_{\perp} \tag{2}$$

where $\Delta h_{\perp}$ is the component of Δh perpendicular to B, so that this torque results in a decrease in the magnitude of $\Delta h_{\perp}$.

The cross-product law generally is not as simple in practice as Eqs. (1) and (2) would indicate. The implementation usually involves torquer coils or bars on three orthogonal spacecraft axes, and different gains may be employed for different axes. In addition, the gains may be large enough that one or more coils are driven to saturation. The limit of infinitely large gains is a bang-bang control law, since the coils are always driven to saturation. Because of these complications, detailed simulations often have been performed to study the effectiveness of momentum unloading systems as a function of control gains, dipole saturation levels, spacecraft mass properties, and orbit parameters. This paper is an outgrowth of such a simulation carried out for SMM.[9] To avoid a time-consuming multiparameter search over various combinations of spacecraft mass and orbit parameters, analytical formulas were derived which allow worst-case conditions to be identified. Detailed simulations are then carried out for these worst-case conditions to establish the adequacy of the momentum management system.

In this paper, the analytical framework of the study is presented first, including a discussion of the approximations involved. Then the orbit averages are performed for the cases of inertially fixed and Earth-pointing spacecraft attitudes. Finally, the specific example of SMM is considered, and the approximate analytical results are compared with the results of dynamics simulations.

Analytical Framework

Two simple variations of the cross-product control law of Eq. (1), the bang-bang control law and the linear control law, are considered in this section. For the bang-bang control law, the commanded dipole along the direction of the ith coil axis is

$$M_{B_i} = C_i \text{sign}(\Delta h \times B)_i \tag{3a}$$

where C_i is the maximum dipole strength in ampere-meters^2, Δh is the excess angular momentum, $\Delta\hat{h}$ is the unit vector in

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the direction of $\boldsymbol{\Delta h}$, and $\boldsymbol{B}$ is the external field. In the coordinate frame defined by the magnetic coils, the vector dipole is

$$M_B = C_1 \frac{(\Delta\hat{h}\times \boldsymbol{B})\cdot\hat{x}}{|(\Delta\hat{h}\times \boldsymbol{B})\cdot\hat{x}|}\hat{x} + C_2 \frac{(\Delta\hat{h}\times \boldsymbol{B})\cdot\hat{y}}{|(\Delta\hat{h}\times \boldsymbol{B})\cdot\hat{y}|}\hat{y} + C_3 \frac{(\Delta\hat{h}\times \boldsymbol{B})\cdot\hat{z}}{|(\Delta\hat{h}\times \boldsymbol{B})\cdot\hat{z}|} \tag{3b}$$

where $\hat{x}$, $\hat{y}$, and $\hat{z}$ are the dipole axes of the three magnetic coils.

For the linear momentum management law, the commanded dipole on the ith axis in response to an excess angular momentum $\boldsymbol{\Delta h}$ is

$$M_{L_i} = g_i(\boldsymbol{\Delta h}\times \boldsymbol{B})_i \tag{4a}$$

where g_i is the gain. It is assumed implicitly that the commandable dipoles are large enough to remain in their linear range of operation. The commanded vector dipole is

$$M_L = g_1[(\boldsymbol{\Delta h}\times \boldsymbol{B})\cdot\hat{x}]\hat{x} + g_2[(\boldsymbol{\Delta h}\times \boldsymbol{B})\cdot\hat{y}]\hat{y} + g_3[(\boldsymbol{\Delta h}\times \boldsymbol{B})\cdot\hat{z}]\hat{z} \tag{4b}$$

The control law effectiveness can be evaluated by comparing the rate at which the magnetic torques remove excess angular momentum to the rate at which disturbance torques add angular momentum. We are principally interested in preventing momentum wheel saturation, so we will consider the case in which the stored angular momentum is near the saturation capacity. The excess angular momentum can always be written as the sum of two terms—a slowly varying term equal to an orbit average evaluated symmetrically about the time of interest and an oscillating term with zero orbit average. It is assumed that the saturation capacity is much larger than the integral of the net disturbance torque over one orbit. In this case, we can neglect the oscillating term in $\boldsymbol{\Delta h}$ compared to the slowly varying part. We can also take the direction of $\boldsymbol{\Delta h}$ to be parallel to the orbit average disturbance torque $\boldsymbol{\Gamma}$; that is, the following approximation can be made:

$$\Delta\hat{h} = \hat{\Gamma} \tag{5}$$

This is the only approximation needed to study the bang-bang law. The linear control law involves the magnitude of $\boldsymbol{\Delta h}$ also. It will be assumed in this case that Δh is equal to some fixed fraction of the momentum capacity of the wheels.

With these approximations in mind, a quantitative estimate of the control law effectiveness is the ratio

$$R = k/\Gamma \tag{6a}$$

where Γ is the magnitude of the secular torque and k is the orbit average of the component of the control torque along the direction of $\hat{\Gamma}$ and $\Delta\hat{h}$,

$$k = \langle |N_c\cdot\Delta\hat{h}|\rangle = \langle |(M\times B)\cdot\Delta\hat{h}|\rangle \tag{6b}$$

where the brackets denote the orbit average. The remainder of this section deals with the computation of k for both the linear and bang-bang laws. Inserting the control dipoles [Eqs. (3b) and (4b)] into Eq. (6b) and evaluating the orbit averages gives the scalars k_B and k_L for the bang-bang and linear laws, respectively. These are

$$k_B = C_1\langle |V_x\cdot B|\rangle + C_2\langle |V_y\cdot B|\rangle + C_3\langle |V_z\cdot B|\rangle \tag{7a}$$

$$k_L = \Delta h\{g_1\langle (V_x\cdot B)^2\rangle + g_2\langle (V_y\cdot B)^2\rangle + g_3\langle (V_z\cdot B)^2\rangle\} \tag{7b}$$

where

$$V_x \equiv \hat{\Gamma}\times\hat{x} \qquad V_y \equiv \hat{\Gamma}\times\hat{y} \qquad V_z \equiv \hat{\Gamma}\times\hat{z} \tag{8}$$

Some further simplifying assumptions must be made to evaluate the integrals implied by Eq. (7). It is assumed that the spacecraft orbit is circular and that the Earth's magnetic field results from a dipole which is fixed relative to the orbit plane. The analytical expression for $\boldsymbol{B}$ is thus[10]

$$B(r) = (a^3H_0/r^3)[3(\hat{m}\cdot\hat{r})\hat{r} - \hat{m}] \tag{9}$$

where $\hat{m}$ is the direction of the dipole, a^3H_0(7.96×10^{15} Weber-meters) is the dipole strength, r is the orbit radius in meters, and $\hat{r}$ is the unit vector from the Earth to the spacecraft.

The *lbn* orbital coordinate system is the most useful for evaluating the orbit averages. In this system, the $\hat{l}$ axis is along the line from the Earth's center to the ascending node, the $\hat{n}$ axis is the orbit normal, and $\hat{b} = \hat{n}\times\hat{l}$. The vectors $\hat{l}$ and $\hat{b}$ define the orbit plane. To simplify the orbit average integrals, $\hat{m}$ is assumed to be fixed relative to the orbit plane; therefore, it is some constant vector. The unit vector giving the spacecraft position in the *lbn* frame is

$$\hat{r} = \begin{bmatrix} \cos\theta \\ \sin\theta \\ 0 \end{bmatrix} \tag{10}$$

where θ is the spacecraft anomaly.

The $\hat{x}$, $\hat{y}$, and $\hat{z}$ axes are fixed with respect to the spacecraft, and thus can be expressed in *lbn* coordinates as a function of θ and the nominal spacecraft attitude. For inertially fixed spacecraft, the vectors in this triad are constant over one orbit. For an Earth-pointing spacecraft, each component of $\hat{x}$, $\hat{y}$, and $\hat{z}$ can be written as a linear sum of $\sin\theta$ and $\cos\theta$.

Orbit Averages for Inertially Fixed Attitude

As an illustration of the application of the algorithm, the orbit averages of the control torque [Eqs. (7) and (8)] are evaluated for an inertially fixed spacecraft. In this case, the $\hat{x}$, $\hat{y}$, and $\hat{z}$ coordinates axes and their cross-product with $\hat{\Gamma}$ are all constant vectors in the *lbn* frame. Expressions are needed for $\langle |B\cdot V|\rangle$ and $\langle (B\cdot V)^2\rangle$, where V is a constant vector. The scalar product can be expressed as a function of θ by using Eqs. (9) and (10):

$$B\cdot V = A\cos2\theta + B\sin2\theta + C \tag{11a}$$

where

$$A = (3/2)(a^3H_0/r^3)(m_1V_1 - m_2V_2) \tag{11b}$$

$$B = (3/2)(a^3H_0/r^3)(m_2V_1 + m_1V_2) \tag{11c}$$

$$C = (1/2)(a^3H_0/r^3)(\hat{m}\cdot V - 3m_3V_3) \tag{11d}$$

where subscripts 1, 2, and 3 indicate components in the *lbn* frame.

The orbit average integrals are

$$\langle |B\cdot V|\rangle = \frac{1}{2\pi}\int_0^{2\pi} |A\cos2\theta + B\sin2\theta + C|\,d\theta$$

$$= \begin{cases} |C| & (|C| > \sqrt{A^2+B^2}) \\ \dfrac{2}{\pi}\sqrt{A^2+B^2}\left[\dfrac{C}{\sqrt{A^2+B^2}}\sin^{-1}\left(\dfrac{C}{\sqrt{A^2+B^2}}\right) + \sqrt{1-\dfrac{C^2}{A^2+B^2}}\right] & (|C| < \sqrt{A^2+B^2}) \end{cases} \tag{12}$$

$$\langle (B\cdot V)^2\rangle = \frac{1}{2\pi}\int_0^{2\pi} (A\cos 2\theta + B\sin 2\theta + C)^2 d\theta$$

$$= \tfrac{1}{2}(A^2+B^2)+C^2 \tag{13}$$

If the orbit normal vector ($\hat{n}=(0,0,1)^T$ in *lbn* coordinates) is introduced, Eqs. (12) and (13) can be expressed in an invariant form as

$$\sqrt{A^2+B^2} = (3/2)(a^3H_0/r^3)\,|\hat{n}\times\hat{m}|\;\;|\hat{n}\times V| \tag{14a}$$

$$|C| = \tfrac{1}{2}\,|B(r\hat{n})\cdot V| \tag{14b}$$

Orbit Averages for Earth-Pointing Attitude

In this section, Eqs. (7a) and (7b) are evaluated for an Earth-pointing spacecraft. A convenient choice for the body axes is $\hat{z}$ along the nadir, $\hat{y}$ along the negative orbit normal, and $\hat{x}=\hat{y}\times\hat{z}$, which is the direction of the velocity for a circular orbit. Expressed in the *lbn* system, these vectors are

$$\hat{x}=\begin{bmatrix}-\sin\theta\\ \cos\theta\\ 0\end{bmatrix}\qquad \hat{y}=\begin{bmatrix}0\\ 0\\ -1\end{bmatrix}\qquad \hat{z}=-\hat{r}=\begin{bmatrix}-\cos\theta\\ -\sin\theta\\ 0\end{bmatrix} \tag{15}$$

The direction for the secular disturbance torque is taken to be the orbit normal. This would be the case if the dominant disturbance were the gravity-gradient torque. The instantaneous value of this torque is

$$N_{GG} = (3\mu/r^3)\,\hat{r}\times I\hat{r} \tag{16}$$

where $\mu = GM_{\rm Earth} = 3.986\times 10^{14}$ m^3/s^2 and I is the spacecraft moment of inertia tensor.

For a nominal Earth-pointing mission $\hat{r}=-\hat{z}$, so in body coordinates the gravity-gradient torque is

$$N_{GG} = (3\mu/r^3)(-I_{23}\hat{x}+I_{13}\hat{y})$$

The orbit-averaged value is found by using Eq. (15) to express N_{GG} as a function of θ and then integrating over one orbit. The result is

$$\Gamma = \langle N_{GG}\rangle = (3\mu/r^3)I_{13}\hat{y} \tag{17}$$

This can be the dominant secular disturbance torque for a spacecraft that is asymmetrical about its pitch ($\hat{y}$) axis. It follows from Eqs. (8) and (17) that

$$V_x = -\hat{z}\qquad V_y = 0\qquad V_z = \hat{x} \tag{18}$$

The scalar products of V with the geomagnetic field vector are evaluated using Eqs. (9) and (18). The values of $\langle |B\cdot V|\rangle$ and $\langle (B\cdot V)^2\rangle$ are evaluated easily using Eqs. (12) and (13), respectively. Inserting these values into Eqs. (7a) and (7b) gives

$$k_B = (2/\pi)(a^3H_0/r^3)(2C_1+C_3)\,|\hat{n}\times\hat{m}| \tag{19}$$

$$k_L = (a^3H_0/r^3)^2\Delta h(2g_1+\tfrac{1}{2}g_3)\,|\hat{n}\times\hat{m}|^2 \tag{20}$$

These expressions give the orbit average of the component of the magnetic control torque along the orbit normal for an Earth-pointing mission. They have been calculated as an illustration of the evaluation of Eq. (7) for an Earth-pointing spacecraft. Note that since $|\hat{n}\times\hat{m}| = \sin i$, where i is the orbit inclination, the average control torque is larger for higher inclinations. The secular disturbance torque is independent of the inclination [Eq. (17)], so the momentum unloading capability is an increasing function of inclination when the gravity-gradient torque dominates the disturbances.

Orbit-Averaged Behavior of the Control Laws for SMM

The SMM spacecraft is approximately axially symmetric, with the symmetry ($\hat{x}$) axis sun-pointing. Thus, the attitude is approximately inertially fixed for one orbit, so Eqs. (12-14) can be used to evaluate the effectiveness of the momentum management after analytical representations of the orbit-averaged disturbance torque and the coil reference axes have been obtained.

The gravity-gradient torque is the dominant disturbance, and its orbit-averaged value is[11]

$$\Gamma = (3\mu/2r^3)(I_{zz}-I_{xx})\,\hat{n}\cdot\hat{S}(\hat{n}\times\hat{S})\ \text{N-m} \tag{21}$$

where I_{xx} and I_{zz} are components of the inertia tensor (approximated as diagonal, with $I_{yy}=I_{zz}$) (kg-m^2), $\hat{S}$ is the sun unit vector, and μ and r are defined in Eqs. (16) and (9), respectively.

The unit vector in the direction of the gravity-gradient torque is

$$\hat{\Gamma} = (\hat{n}\times\hat{S})\,/\,|\hat{n}\times\hat{S}| \tag{22}$$

The coil axes in inertial coordinates must be known. A convenient system which can be considered to be inertially fixed over one orbit is the *S-U-N* frame. This is defined with its x axis along $\hat{S}$, the apparent spacecraft-to-sun unit vector. The $\hat{U}$ and $\hat{N}$ axes are defined by

$$\hat{U} = (\hat{P}\times\hat{S})\,/\,|\hat{P}\times\hat{S}|\qquad \hat{N}=\hat{S}\times\hat{U} \tag{23}$$

where $\hat{P}$ is the inertially fixed solar spin axis.

The magnetic coils are in the Modular Attitude Control System (MACS) reference frame. For the nominal SMM attitude, the MACS and *S-U-N* systems are related by a 150-deg rotation about $\hat{S}$. The MACS coordinate axes in terms of $\hat{S}$, $\hat{U}$, and $\hat{N}$ are

$$\hat{x}=\hat{S}\qquad \hat{y}=-\frac{\sqrt{3}}{2}\hat{U}+\frac{1}{2}\hat{N}\qquad \hat{z}=-\frac{1}{2}\hat{U}-\frac{\sqrt{3}}{2}\hat{N} \tag{24}$$

Substituting Eqs. (22) and (24) into Eq. (8) gives

$$V_x=\frac{(\hat{n}\times\hat{S})\times\hat{S}}{|\hat{n}\times\hat{S}|}\equiv\hat{W}\qquad V_y=H_y\hat{S}\qquad V_z=H_z\hat{S} \tag{25}$$

where

$$H_y\equiv\frac{-\frac{\sqrt{3}}{2}\hat{n}\cdot\hat{U}+\frac{1}{2}\hat{n}\cdot\hat{N}}{|\hat{n}\times\hat{S}|}\qquad H_z\equiv\frac{-\frac{1}{2}\hat{n}\cdot\hat{U}-\frac{\sqrt{3}}{2}\hat{n}\cdot\hat{N}}{|\hat{n}\times\hat{S}|} \tag{26}$$

Note that $H_y^2+H_z^2=1$.

Inserting Eqs. (25) into Eq. (7a) gives, for the bang-bang law,

$$k_B = C_1\langle |B\cdot\hat{W}|\rangle + \{C_2|H_y|+C_3|H_z|\}\langle |B\cdot\hat{S}|\rangle \tag{27}$$

The orbit average integral is given by Eq. (12). The quantities $\sqrt{A^2+B^2}$ and $|C|$ [Eqs. (14a) and (14b)] for $V=\hat{S}$ and $V=\hat{W}$ are

$$\sqrt{A_w^2+B_w^2} = (3/2)(a^3H_0/r^3)\,|\hat{n}\times\hat{m}|\,|\hat{n}\cdot\hat{S}| \tag{28a}$$

$$|C_w| = \frac{1}{2}\left(\frac{a^2H_0}{r^3}\right)\left|\frac{(\hat{n}\cdot\hat{m})[3(\hat{n}\cdot\hat{S})^2-2]-(\hat{n}\cdot\hat{S})(\hat{S}\cdot\hat{m})}{|\hat{n}\times\hat{S}|}\right| \tag{28b}$$

$$\sqrt{A_s^2+B_s^2} = (3/2)(a^3H_0/r^3)\,|\hat{n}\times\hat{m}|\,|\hat{n}\times\hat{S}| \tag{28c}$$

$$|C_s| = (1/2)(a^3H_0/r^3)\,|3(\hat{n}\cdot\hat{m})(\hat{n}\cdot\hat{S}) - \hat{m}\cdot\hat{S}| \tag{28d}$$

Equations (12), (27), and (28) define k_B in terms of the vectors $\hat{n}$, $\hat{m}$, $\hat{S}$, $\hat{U}$, and $\hat{N}$. These are all assumed to be inertially fixed for one orbit. The resulting equations simplify in the special case that the orbit normal is aligned with the Earth's dipole ($\hat{n}=\hat{m}$). In this case, Eq. (27) becomes

$$k_B = (a^3H_0/r^3)[C_1|\hat{n}\times\hat{S}| + (C_2|H_y| + C_3|H_z|)|\hat{n}\cdot\hat{S}|] \quad (\hat{n} \text{ parallel to } \hat{m}) \tag{29}$$

An invariant expression for k_L can also be obtained. Inserting Eqs. (25) into Eq. (7b) gives

$$k_L = \Delta h\{g_1\langle(B\cdot\hat{W})^2\rangle + (g_2H_y^2+g_3H_z^2)\langle(B\cdot\hat{S})^2\rangle\} \tag{30}$$

The orbit average integrals are evaluated using Eqs. (13) and (28). The result can be written

$$k_L = \left(\frac{a^3H_0}{r^3}\right)^2 \Delta h\left\{g_1\left[1+\frac{1}{8}|\hat{n}\times\hat{m}|^2 + \left(\frac{\hat{m}\cdot\hat{S}-(\hat{n}\cdot\hat{m})(\hat{n}\cdot\hat{S})}{2|\hat{n}\times\hat{S}|}\right)^2\right] + \tfrac{1}{4}(g_2H_y^2+g_3H_z^2-g_1) \times\left[(3(\hat{n}\cdot\hat{m})\ (\hat{n}\cdot\hat{S}) - \hat{m}\cdot\hat{S})^2 + (9/2)|\hat{n}\times\hat{m}|^2|\hat{n}\times\hat{S}|^2\right]\right\} \tag{31}$$

If $g_1=g_2=g_3\equiv g$, Eq. (31) simplifies to

$$k_L = \left(\frac{a^3H_0}{r^3}\right)^2 g\Delta h\left\{1+\frac{1}{8}|\hat{n}\times\hat{m}|^2 + \left[\frac{\hat{m}\cdot\hat{S}-(\hat{n}\cdot\hat{m})(\hat{n}\cdot\hat{S})}{2|\hat{n}\times\hat{S}|}\right]^2\right\} \tag{32}$$

If the orbit normal is aligned with the Earth's dipole, Eq. (31) gives

$$k_L = (a^3H_0/r^3)^2\Delta h[g_1|\hat{n}\times\hat{S}|^2 + (g_2H_y^2+g_3H_z^2)(\hat{n}\cdot\hat{S})^2] \quad (\hat{n} \text{ parallel to } \hat{m}) \tag{33}$$

Quantitative Results for SMM

The effectiveness of the control laws can be quantified by plotting the ratio R of the orbit average of the component of the control torque along the direction of the secular disturbance to the magnitude of the secular torque. If R is greater than 1, then the component of the control torque along $\hat{\Gamma}$ is larger than Γ, and the control laws should be adequate to prevent wheel saturation.

The value of the ratio depends on the control law gains, the spacecraft moments of inertia [Eq. (21)], and scalar products involving the vectors $\hat{n}$, $\hat{m}$, $\hat{S}$, $\hat{U}$, and $\hat{N}$. (Note that both k_B and Γ are proportional to the inverse cube of the orbit radius, so the ratio R is independent of this parameter. This also is true for the linear law if the gains g are scaled properly.) Since these vectors are all assumed to be inertially fixed for one orbit, the geocentric inertial (GCI) coordinate system is used to evaluate them. The unit vector in the direction of the Earth's dipole moment, $\hat{m}$, is chosen to lie along the Earth's spin axis (the z axis of GCI). This dipole is actually fixed on the Earth at an angle of 12 deg to the spin axis, so it precesses about the z axis with a period of 1 day. The fixed dipole approximation, therefore, amounts to replacing the inertial magnetic field by a constant average dipole field. In GCI coordinates, the $\hat{S}$-$\hat{U}$-$\hat{N}$ coordinate axes depend on the time of year [see Eq. (23)], and the orbit normal $\hat{n}$ is a function of the right ascension of the ascending node (Ω) and the inclination (i). Therefore, the ratio R depends on the control law gains, the time of year, and the orbit parameters Ω and i. In the following, the ratio R is considered as a function of i only, taking the time of year and value of Ω that give the smallest R, since i is essentially constant but Ω precesses because of the oblateness of the Earth for near-Earth orbits.

For the linear control law with equal gains on all three axes, it is possible to proceed analytically. Equation (21) gives

$$\Gamma = \tfrac{3}{4}(\mu/r^3)(I_{zz}-I_{xx})\sin 2\psi \tag{34}$$

and Eq. (32) gives

$$k_L = (a^3H_0/r^3)^2 g\Delta h[1+\tfrac{1}{8}\sin^2 i(1+2\cos^2\Phi)] \tag{35}$$

where the orbit inclination i, the sun-orbit normal angle ψ, and the rotation angle Φ are shown in Fig. 1. The obliquity of the ecliptic e is equal to 23.44 deg. The angle α between $\hat{m}$ and $\hat{S}$ is in the range 90 deg $- e$ to 90 deg $+ e$. Only inclinations less than 90 deg will be considered; the value of R is symmetrical about 90-deg inclinations. For the smallest R, Γ is maximized by taking ψ as close to 45 deg as possible and then k_L is minimized by taking Φ as close to 90 deg as is consistent with this value of ψ. The minimizing conditions are

$$\psi = 90\text{ deg} - e - i \quad \text{for} \quad i \le 45\text{ deg} - e \tag{36a}$$

$$\psi = 45\text{ deg} \quad \text{for} \quad i \ge 45\text{ deg} - e \tag{36b}$$

$$\alpha = 90\text{ deg} - e \quad \text{for} \quad i \le \arccos(\sqrt{2}\sin e) \tag{36c}$$

$$\Phi = 90\text{ deg} \quad \text{for} \quad i \ge \arccos(\sqrt{2}\sin e) \tag{36d}$$

which result in

$$R = K\frac{1+\tfrac{3}{8}\sin^2 i}{\sin 2(e+i)} \quad \text{for} \quad i \le 45\text{ deg} - e \tag{37a}$$

$$= K[1+\tfrac{1}{8}\sin^2 i + \tfrac{1}{4}(\sqrt{2}\sin e - \cos i)^2] \quad \text{for } 45\text{ deg} - e \le i \le \arccos(\sqrt{2}\sin e) \tag{37b}$$

$$= K(1+\tfrac{1}{8}\sin^2 i) \quad \text{for} \quad i \ge \arccos(\sqrt{2}\sin e) \tag{37c}$$

where

$$K \equiv \frac{4}{3}\left(\frac{a^3H_0}{r^3}\right)^2 \frac{r^3 g\Delta h}{\mu(I_{zz}-I_{xx})} \tag{37d}$$

The ratio R/K is plotted as a function of i Fig. 2. The rapid decrease in R for small inclinations is due to increasing Γ. The slow increase in R for inclinations above 18 deg is due to increasing k_L.

The SMM orbit inclination will be in the range 28.5-33 deg, which gives $R=1.06K$. An SMM momentum management study[9] was performed for moments of inertia $I_{zz}=3982$ kg-m^2, and $I_{xx}=1818$ kg-m^2 and an orbit radius of 6953 km. It was found that the linear control law was insufficient for gains less than 2×10^5 A-m^2/(N-m-s-T) but was adequate for larger gains. Putting this critical gain into Eq. (37d) gives $K=0.058\ \Delta h$ and $R=0.062\ \Delta h$ with Δh in N-m-s. Thus, the analytical results agree with the detailed simulations (that is, $R=1$ implies marginal control capability) if $\Delta h=16$ N-m-s. This is 80% of the saturation capacity of one wheel. The detailed simulations[9] show that in regions where wheel saturation is marginally avoided, the wheel speeds oscillate about a value which is 80% of their maximum values, and the additional 20% leading to saturation is due to the fluctuating

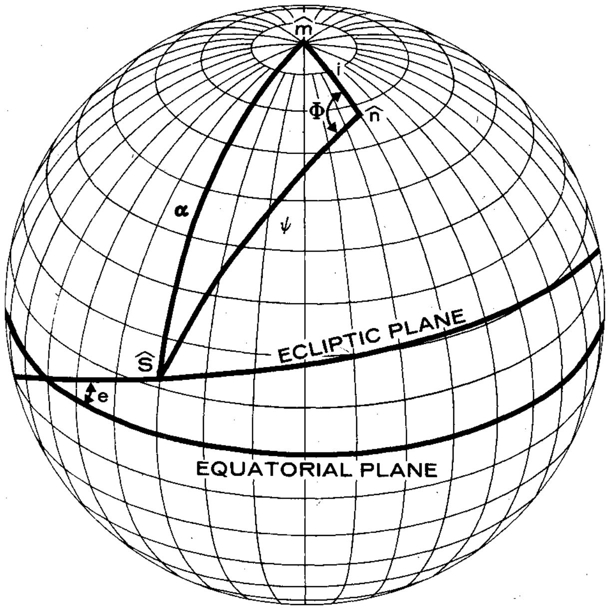

Fig. 1 Geometry for linear control law for an axially symmetric sun-pointing spacecraft. $\hat{S}$ is the spacecraft-to-sun unit vector, $\hat{n}$ is the orbit normal, and $\hat{m}$ is the Earth's north pole. The curves are arcs of great circles.

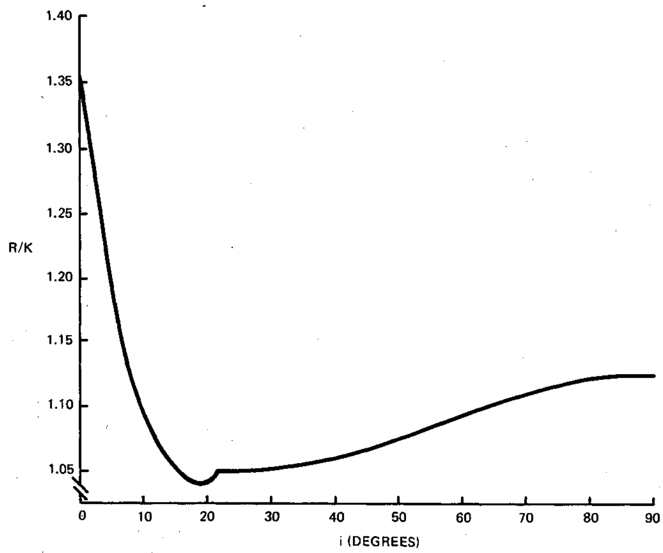

Fig. 2 Performance index of linear control law vs orbit inclination (i) for an axially symmetric sun-pointing spacecraft.

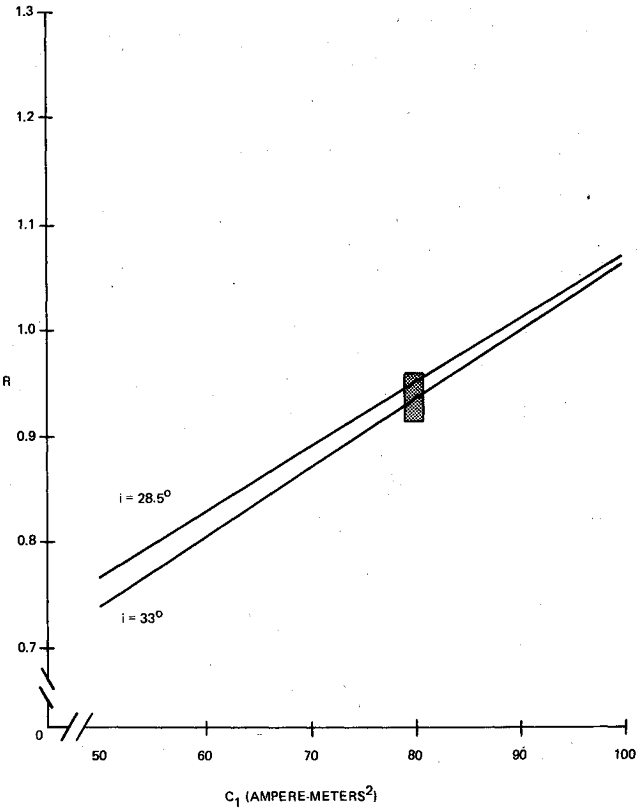

Fig. 3 Control-to-disturbance ratio for the bang-bang magnetic control law as a function of x axis coil strength. The shaded region marks the area where the simulator predicts the control law to be marginally effective.

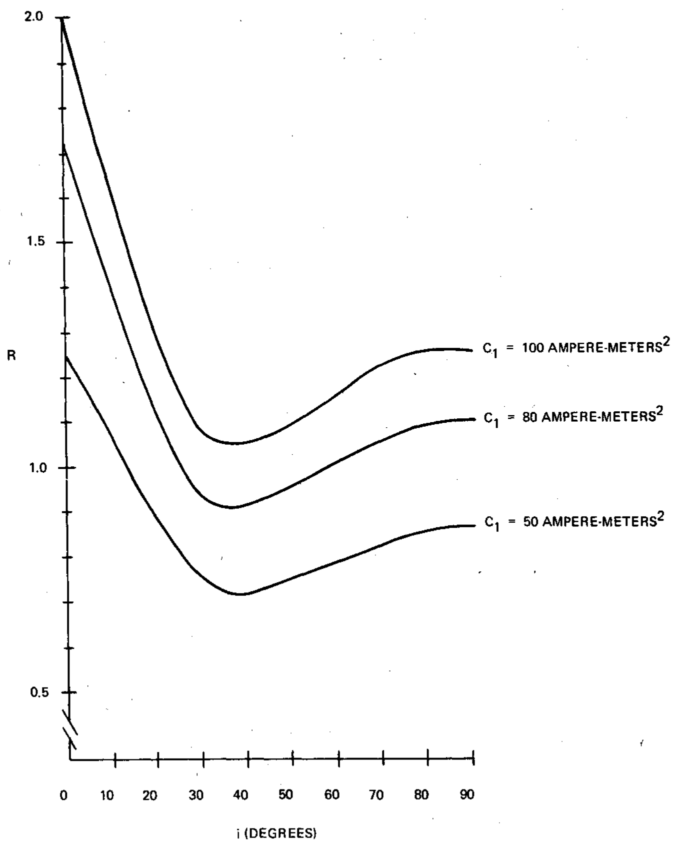

Fig. 4 Control-to-disturbance ratio for the bang-bang magnetic control law as a function of orbit inclination ($C_2 = C_3 = 50$ A-m^2).

component of the instantaneous gravity-gradient torque. Thus, it appears empirically that Δh should be taken to be approximately 80% of the reaction wheel capacity on one axis to test the adequacy of momentum unloading with the linear control law.

The bang-bang control law is not as amenable to analytical treatment, so a numerical analysis of the predictions of Eqs. (12), (27), and (28) is required. Again, a worst-case orbit for each inclination is found by maximizing Γ, but the worst-case geometry for the bang-bang law also requires that Γ lie along one of the torque coil directions,[12] rendering that torque ineffective for momentum unloading. The momentum management study[9] concluded that the bang-bang magnetic control law could not prevent momentum wheel saturation if the three MACS coils had a dipole capacity of 50 A-m^2, a single-point failure mode with the capacity originally planned for SMM. The improvement in performance due to increasing the x-coil strength was studied, since the x-axis coil is the most important for SMM mass properties and low-orbit inclination

[Eqs. (29) and (33)]. It was found that control would be adequate if the x, y, and z coils had capacities of 100, 50, and 50 A-m^2, respectively. Therefore, to see if the control-to-disturbance ratio R accurately predicts control law performance, R is plotted in Fig. 3 as a function of x-axis coil strength for values greater than 50 A-m^2. The inclination is fixed, and y and z coil strengths are 50 A-m^2. Also indicated on the graph is the region of minimum coil strength that will prevent wheel saturation as determined by a detailed attitude dynamics simulator. This value is about 80 A-m^2. This figure shows that the ratio underestimates the control system effectiveness by about 5-10%. That is, the control is expected to be inadequate if the ratio is less than 1, but the simulator shows that the critical value is in the range 0.9-0.95.

About half of this discrepancy can be attributed to the fixed dipole model of the magnetic field. The simulator normally uses an eighth-order IGRF magnetic field model. However, when the simulator is run using a second-order field (a rotating dipole), the critical x-axis coil strength becomes 85 A-m^2, moving the critical ratio closer to 1. The remainder of the discrepancy is probably due to the oscillating component of $\Delta\hat{h}$, which has been ignored in this discussion.

In Fig. 4, the control-to-disturbance ratio is plotted as a function of orbit inclination for various values of x-axis dipole strength for the bang-bang law. This figure shows that the bang-bang control law should be most effective for an equatorial orbit, get worse at the middle latitudes, and then become more effective toward the pole. This behavior is similar to that of the linear control law (Fig. 2).

Conclusions

The analytic formulas given by Eqs. (12), (27), and (28) for the bang-bang case or by Eq. (31) for the linear case can be used for mission planning to establish the size of magnetic torquers and the gains needed for a given mission with inertially fixed attitude. Although the specific example of a sun-pointing mission was treated, generalization to a stellar-pointing mission is straightforward. The results of Eqs. (19) and (20) can be used for Earth-pointing missions where the gravity-gradient is the dominant secular disturbance torque. The general approach presented in this paper will be useful for any case in which an analytical model of the secular disturbance torque is available.

The analytic results should be good to about 10%, as was seen in this study. If more accurate results are desired, detailed simulations can be performed. However, only critical cases indicated by the analytic formulas presented here need be studied in detail, avoiding a time-consuming multiple-parameter search in the simulations.

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